THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Three realizations of quantum affine algebra Uq(A(2)2)

Citation for published version:
Shapiro, A 2010, 'Three realizations of quantum affine algebra Uq(A(2)2)', *Theoretical and mathematical physics*, vol. 165, no. 2, pp. 217–232. https://doi.org/10.4213/tmf6572

Digital Object Identifier (DOI):
10.4213/tmf6572

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Theoretical and mathematical physics

General rights
Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy
The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.

OPEN ACCESS

Download date: 07. May. 2023

ITEP-TH-65/09

Three realizations of quantum affine algebra $U_q(A_2^{(2)})$

A. Shapiro⋆[1]

⋆*Institute of Theoretical & Experimental Physics, 117259 Moscow, Russia*

arXiv:1009.1932v2 [math.QA] 27 Nov 2010

Abstract

We establish explicit isomorphisms between three realizations of the quantum twisted affine algebra $U_q(A_2^{(2)})$: the Drinfeld ("current") realization, the Chevalley realization, and the so-called RLL realization, investigated by Faddeev, Reshetikhin and Takhtajan.

0 Introduction

There exist just two quantum affine algebras of rank 2: $U_q(\widehat{\mathfrak{sl}}_2)$ and $U_q(A_2^{(2)})$. The algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and its representation theory are both very well studied. The situation with $U_q(A_2^{(2)})$ is somewhat different, although the algebra itself and its representation are interesting from the standpoints of both mathematics and mathematical physics.

The algebra $U_q(A_2^{(2)})$ first appeared in works on physics. The fundamental representation of the $\mathcal{R}$-matrix of $U_q(A_2^{(2)})$ was obtained in [IK] as the $\mathcal{R}$-matrix of the quantum version of the Shabat-Mikhailov model, also known as the Izergin-Korepin model. The same representation was later obtained in [J] among fundamental representations of $\mathcal{R}$-matrices of other nonexceptional affine Lie algebras. The Bethe ansatz technique was later extended onto $U_q(A_2^{(2)})$ in [T]. Finally, the ideas of the thermodynamic Bethe ansatz for $U_q(A_2^{(2)})$ were developed in [FRS], and some finite dimensional representations were obtained there.

The algebra $U_q(A_2^{(2)})$ was also investigated from an algebraic standpoint. In [KT], the Cartan-Weyl basis for $U_q(A_2^{(2)})$ was established, and the universal $\mathcal{R}$-matrix was written in terms of infinite products of elements of the Cartan-Weyl basis. A classification of finite dimensional representations of algebra $U_q(A_2^{(2)})$ was obtained in [CP] by means of the Drinfeld polynomials. Later, an integral formula for the universal $\mathcal{R}$-matrix for $U_q(A_2^{(2)})$ appeared in [DK], where $U_q(A_2^{(2)})$ was treated as a topological Hopf algebra with the Drinfeld comultiplication. In the same work, the Serre relations in $U_q(A_2^{(2)})$ were represented in terms of zeros and poles of products of the Drinfeld currents. Finally, integral representations for factors of the universal $\mathcal{R}$-matrix for $U_q(A_2^{(2)})$ were derived in [KS], where $U_q(A_2^{(2)})$ was endowed with the standard Hopf structure.

Quantum affine algebras allow three different realizations with different Hopf structures. The first one is the "standard" realization given by the Chevalley generators and relations determined by the corresponding Cartan matrix. The standard realization has a small number of generators, but is unfortunately very difficult to use in applications. The second realization, the Drinfeld

[1]E-mail: alexander.m.shapiro@gmail.com

"new realization", was first established in [D] by means of generating functions (the Drinfeld "currents") and relations on them. The Drinfeld realization allows using methods of complex analysis. Moreover, it makes it possible to give the classification of the finite dimensional representations of the quantum affine algebra. Finally, the third one is the RLL realization, based on the Faddeev-Reshetikhin-Takhtajan-Semenov-Tian-Shansky approach, where generators are combined into L-operators, satisfying the famous Yang-Baxter equation (see [FRT], [RS]). The simplicity of comultiplication in the RLL realization allows to construct new representations as tensor products of already known ones. The RLL realization is therefore widely used in physical models.

It is universally acknowledged that the three realizations are isomorphic, although precise proofs hardly exist for any algebra other then of the $U_q(\widehat{s}l_n)$ type. For $U_q(\widehat{s}l_n)$, an isomorphism between the standard and the Drinfeld realizations was established in [DF], whereas links between the Drinfeld and the RLL approaches were studied in [DK2].

In the case of quantum twisted affine algebras, the RLL realization requires additional relations. Although it is believed that the three realizations are also isomorphic for the twisted algebras, there is no full understanding of what the exact isomorphism should look like. An isomorphism between the standard and the Drinfeld realizations for $U_q(A_2^{(2)})$ was established in [KS], and an isomorphism between the RLL and the Drinfeld realizations was partly derived in [YZ]. A more complete bibliography can be found in [H].

In this work, we obtain a full description of the three realizations of algebra $U_q(A_2^{(2)})$ (without grading element and with zero central charge), the isomorphisms between them, and the links between the three Hopf structures. We also pay special attention to the following fact: each realization has a "minimal" set of generators (or almost minimal in the twisted case) and an extended set of generators. These two sets should be linked by an analogue of the PBW-theorem if there exists one. In the standard realization, these sets are the Chevalley generators and the Cartan-Weyl basis. In the Drinfeld realization, they are the Drinfeld currents and the so-called "composite currents" (see [DK3]). Finally, in the RLL-realization, the Gaussian coordinates immediately above or below the diagonal form the "minimal" set, and all the Gaussian coordinates form the extended set. We note that these extended sets of generators are crucial for calculations of the universal weight function. Here, we obtain a link between projections of the composite currents and the Gaussian coordinates for $U_q(A_2^{(2)})$ as was done for $U_q(\widehat{\mathfrak{gl}}_n)$ in [KP].

The paper is organized as follows. In the first section we introduce the three realizations. In Section 2, we obtain the isomorphisms between them. In Section 3, we represent elements of the extended set of Gaussian coordinates in terms of composite currents.

1 Realizations of $U_q(A_2^{(2)})$

For simplicity of exposition, we consider the algebra $U_q(A_2^{(2)})$ without grading element and with zero central charge. All the statements also hold for an algebra with grading element and arbitrary central charge, but the formulas become more involved.

1.1 Drinfeld realization

Let $\mathcal{A}_\mathcal{D}$ be the associative algebra generated by elements

$$e_n, f_n,\ n \in \mathbb{Z}, \quad a_n,\ n \in \mathbb{Z}\setminus\{0\}\,, \quad \text{and} \quad k^{\pm 1},$$

subject to certain commutation relations. The relations are given as formal power series identities for generating functions (currents)

$$e(z) = \sum_{k \in \mathbb{Z}} e_k z^{-k}, \quad f(z) = \sum_{k \in \mathbb{Z}} f_k z^{-k}, \quad K^{\pm}(z) = k^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{n>0} a_{\pm n} z^{\mp n} \right)$$

as follows:

$$\begin{aligned}
(z - q^2 w)(qz + w)e(z)e(w) &= (q^2 z - w)(z + qw)e(w)e(z), \\
(q^2 z - w)(z + qw)f(z)f(w) &= (z - q^2 w)(qz + w)f(w)f(z), \\
K^+(z)e(w)K^+(z)^{-1} &= \alpha(w/z)e(w), \\
K^+(z)f(w)K^+(z)^{-1} &= \alpha(w/z)^{-1}f(w), \\
K^-(z)e(w)K^-(z)^{-1} &= \alpha(z/w)^{-1}e(w), \\
K^-(z)f(w)K^-(z)^{-1} &= \alpha(z/w)f(w), \\
K^{\pm}(z)K^{\pm}(w) &= K^{\pm}(w)K^{\pm}(z), \\
K^-(z)K^+(w) &= K^+(w)K^-(z), \\
e(z)f(w) - f(w)e(z) &= \frac{1}{q - q^{-1}} \left(\delta(z/w)K^+(w) - \delta(z/w)K^-(z) \right).
\end{aligned}$$

where

$$\alpha(x) = \frac{(q^2 - x)(q^{-1} + x)}{(1 - q^2 x)(1 + q^{-1}x)},$$

and $\delta\left(\frac{z}{w}\right)$ is a formal Laurent series, given by

$$\delta(z/w) = \sum_{n \in \mathbb{Z}} (z/w)^n.$$

The generating functions $e(z)$ and $f(z)$ also satisfy the cubic Serre relations (see [D]):

$$\begin{aligned}
\underset{z_1,z_2,z_3}{\mathrm{Sym}} \left(q^{-3}z_1 - (q^{-2} + q^{-1})z_2 + z_3 \right) e(z_1)e(z_2)e(z_3) &= 0, \\
\underset{z_1,z_2,z_3}{\mathrm{Sym}} \left(q^{-3}z_1^{-1} - (q^{-2} + q^{-1})z_2^{-1} + z_3^{-1} \right) f(z_1)f(z_2)f(z_3) &= 0, \\
\underset{z_1,z_2,z_3}{\mathrm{Sym}} \left(q^{3}z_1^{-1} - (q^{2} + q)z_2^{-1} + z_3^{-1} \right) e(z_1)e(z_2)e(z_3) &= 0, \\
\underset{z_1,z_2,z_3}{\mathrm{Sym}} \left(q^{3}z_1 - (q^{2} + q)z_2 + z_3 \right) f(z_1)f(z_2)f(z_3) &= 0.
\end{aligned}$$

The Hopf algebra structure on $\mathcal{A}_{\mathcal{D}}$ can be defined as follows:

$$\begin{aligned}
\Delta_{\mathcal{D}}(e(z)) &= e(z) \otimes 1 + K^-(z) \otimes e(z), \\
\Delta_{\mathcal{D}}(f(z)) &= 1 \otimes f(z) + f(z) \otimes K^+(z), \\
\Delta_{\mathcal{D}}(K^{\pm}(z)) &= K^{\pm}(z) \otimes K^{\pm}(z), \\
S_{\mathcal{D}}(e(z)) &= -(K^-(z))^{-1}e(z), \\
S_{\mathcal{D}}(f(z)) &= -f(z)(K^+(z))^{-1}, \\
S_{\mathcal{D}}(K^{\pm}(z)) &= (K^{\pm}(z))^{-1}, \\
\varepsilon_{\mathcal{D}}(e(z)) &= 0, \\
\varepsilon_{\mathcal{D}}(f(z)) &= 0, \\
\varepsilon_{\mathcal{D}}(K^{\pm}(z)) &= 1.
\end{aligned}$$

where $\Delta_{\mathcal{D}}$, $\varepsilon_{\mathcal{D}}$ and $S_{\mathcal{D}}$ are the comultiplication, the counit and the antipode maps, respectively. We call $\Delta_{\mathcal{D}}$ a *Drinfeld comultiplication*. Here, we must note that $\mathcal{A}_{\mathcal{D}}$ is a topological bialgebra and $\Delta_{\mathcal{D}}$ is a map from $\mathcal{A}_{\mathcal{D}}$ to a topological extension of its tensor square (see [EKP, Section 2] for details).

1.2 Chevalley realization

Another realization of $U_q(A_2^{(2)})$ is given in terms of Chevalley generators. Let the algebra $\mathcal{A}_{\mathcal{C}h}$ be the associative algebra generated by elements $e_{\pm\alpha}$, $e_{\pm(\delta-2\alpha)}$, $k_\alpha^{\pm1}$, $k_{\delta-2\alpha}^{\pm1}$, satisfying the following relations:

$$\begin{aligned}
k_\alpha e_{\pm\alpha} k_\alpha^{-1} &= q^{\pm1} e_{\pm\alpha}, & k_\alpha e_{\pm(\delta-2\alpha)} k_\alpha^{-1} &= q^{\mp2} e_{\pm(\delta-2\alpha)},\\
k_{\delta-2\alpha} e_{\pm\alpha} k_{\delta-2\alpha}^{-1} &= q^{\mp2} e_{\pm\alpha}, & k_{\delta-2\alpha} e_{\pm(\delta-2\alpha)} k_{\delta-2\alpha}^{-1} &= q^{\pm4} e_{\pm(\delta-2\alpha)},\\
k_\alpha^2 k_{\delta-2\alpha} &= 1, & \left[e_{\pm\alpha}, e_{\mp(\delta-2\alpha)}\right] &= 0,\\
[e_\alpha, e_{-\alpha}] &= \frac{k_\alpha - k_\alpha^{-1}}{q-q^{-1}}, & \left[e_{\delta-2\alpha}, e_{-(\delta-2\alpha)}\right] &= \frac{k_{\delta-2\alpha} - k_{\delta-2\alpha}^{-1}}{q-q^{-1}},\\
(\mathrm{ad}_q\, e_{\pm\alpha})^5 e_{\pm(\delta-2\alpha)} &= 0, & (\mathrm{ad}_q\, e_{\pm(\delta-2\alpha)})^2 e_{\pm\alpha} &= 0,
\end{aligned}$$

where

$$\begin{aligned}
(\mathrm{ad}_q\, e_{\pm\alpha})(x) &= e_{\pm\alpha} x - k_\alpha^{\pm1} x k_\alpha^{\mp1} e_{\pm\alpha},\\
(\mathrm{ad}_q\, e_{\pm(\delta-2\alpha)})(x) &= e_{\pm(\delta-2\alpha)} x - k_{\delta-2\alpha}^{\pm1} x k_{\delta-2\alpha}^{\mp1} e_{\pm(\delta-2\alpha)}.
\end{aligned}$$

The Hopf algebra structure associated with this realization can be defined by

$$\begin{aligned}
\Delta(e_\alpha) &= e_\alpha \otimes 1 + k_\alpha \otimes e_\alpha, & \Delta(e_{\delta-2\alpha}) &= e_{\delta-2\alpha} \otimes 1 + k_{\delta-2\alpha} \otimes e_{\delta-2\alpha},\\
\Delta(e_{-\alpha}) &= 1 \otimes e_{-\alpha} + e_{-\alpha} \otimes k_\alpha^{-1}, & \Delta(e_{-(\delta-2\alpha)}) &= 1 \otimes e_{-(\delta-2\alpha)} + e_{-(\delta-2\alpha)} \otimes k_{\delta-2\alpha}^{-1},\\
\Delta(k_\alpha) &= k_\alpha \otimes k_\alpha, & \Delta(k_{\delta-2\alpha}) &= k_{\delta-2\alpha} \otimes k_{\delta-2\alpha},\\
\varepsilon(e_{\pm\alpha}) &= 0, & \varepsilon(e_{\pm(\delta-2\alpha)}) &= 0,\\
\varepsilon(k_\alpha^{\pm1}) &= 1, & \varepsilon(k_{\delta-2\alpha}^{\pm1}) &= 1,\\
S(e_\alpha) &= -k_\alpha^{-1} e_\alpha, & S(e_{\delta-2\alpha}) &= -k_{\delta-2\alpha}^{-1} e_{\delta-2\alpha},\\
S(e_{-\alpha}) &= -e_{-\alpha} k_\alpha, & S(e_{-(\delta-2\alpha)}) &= -e_{-(\delta-2\alpha)} k_{\delta-2\alpha},\\
S(k_\alpha^{\pm1}) &= k_\alpha^{\mp1}, & S(k_{\delta-2\alpha}^{\pm1}) &= k_{\delta-2\alpha}^{\mp1},
\end{aligned}$$

where Δ, ε and S are the comultiplication, the counit and the antipode maps, respectively. We call Δ the *standard comultiplication*. We also consider the opposite comultiplication

$$\Delta^{op} = \sigma \circ \Delta, \qquad \text{where} \qquad \sigma(u \otimes v) = v \otimes u.$$

Now, let us recall the construction of the Cartan-Weyl basis, obtained for $U_q(A_2^{(2)})$ in [KT][2]. The twisted affine algebra $A_2^{(2)}$ has the root system $\Delta = \Delta_+ \cup \Delta_-$, where

$$\Delta_+ = \alpha \cup \{n\delta,\ \pm\alpha + n\delta,\ \pm 2\alpha + (2n-1)\delta \mid n \in \mathbb{N}\}, \qquad \text{and} \qquad \Delta_- = -\Delta_+.$$

[2] To adapt the results of [KT] to our case, one should replace q-commutators with q^{-1}-commutators in the construction of the Cartan-Weyl basis.

The roots $\gamma \in \Delta_+$ are called *positive*, the roots $\gamma \in \{n\delta,\ n \in \mathbb{Z}\}$ are called *imaginary*, and the roots $\gamma \in \Delta \setminus \{n\delta,\ n \in \mathbb{Z}\}$ are called *real*. We define a scalar product on Δ by

$$(\alpha, \alpha) = 1, \qquad (\alpha, \delta) = (\delta, \delta) = 0.$$

Now, let

$$q_\gamma = q^{(\gamma,\gamma)}, \qquad (a)_q = \frac{q^a - 1}{q - 1}, \qquad (n)_q! = (1)_q \cdot \dots \cdot (n)_q,$$

and

$$\exp_q(x) = \sum_{n \geqslant 0} \frac{x^n}{(n)_q!} = 1 + x + \frac{x^2}{(2)_q!} + \dots .$$

We fix the normal ordering on Δ

$$\alpha,\ 2\alpha + \delta,\ \alpha + \delta,\ 2\alpha + 3\delta,\ \alpha + 2\delta,\ \dots,\ \delta,\ 2\delta,\ 3\delta,\ \dots,\ 2\delta - \alpha,\ 3\delta - 2\alpha,\ \delta - \alpha,\ \delta - 2\alpha.$$

In accordance with the procedure of the construction of the Cartan-Weyl basis described in [KT], we set

$$\begin{aligned}
&e_{\delta-\alpha} = a[e_\alpha, e_{\delta-2\alpha}]_{q^{-1}}, \qquad e'_\delta = b[e_\alpha, e_{\delta-\alpha}]_{q^{-1}}, \qquad e_{\alpha+n\delta} = b[e_{\alpha+(n-1)\delta}, e'_\delta]_{q^{-1}}, \\
&e_{\delta-\alpha+n\delta} = b[e'_\delta, e_{\delta-\alpha+(n-1)\delta}]_{q^{-1}}, \qquad e'_{n\delta} = b[e_{\alpha+(n-1)\delta}, e_{\delta-\alpha}]_{q^{-1}}, \\
&e_{2\alpha+(2n+1)\delta} = a[e_{\alpha+n\delta}, e_{\alpha+(n+1)\delta}]_{q^{-1}}, \qquad e_{\delta-2\alpha+2n\delta} = a[e_{\delta-\alpha+n\delta}, e_{\delta-\alpha+(n-1)\delta}]_{q^{-1}},
\end{aligned} \tag{1.1}$$

where

$$a = \frac{1}{\sqrt{q + q^{-1}}}, \qquad b = \frac{1}{\sqrt{q + 1 + q^{-1}}},$$

and

$$[e_\beta, e_\gamma]_{q^{-1}} = e_\beta e_\gamma - q^{-(\beta,\gamma)} e_\gamma e_\beta.$$

holds for all $\beta, \gamma \in \Delta$. Finally, we define *imaginary* roots $e_{n\delta}$ by means of the Schur polynomials:

$$e'_{n\delta} = \sum_{p_1 + 2p_2 + \dots + np_n = n} \frac{\left((q - q^{-1})b^{-1}\right)^{\sum p_i - 1}}{p_1! \dots p_n!} e_\delta^{p_1} e_{2\delta}^{p_2} \dots e_{n\delta}^{p_n}. \tag{1.2}$$

Monomials in the elements $e_{n\delta}$, $e_{n\delta \pm \alpha}$ and $k_\alpha^{\pm 1}$ form a linear basis of $\mathcal{A}_{\mathcal{C}h}$.

1.3 RLL realization

Let

$$R(x) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & e & 0 & p & 0 & 0 \\
0 & f & 0 & a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & g & 0 & c & 0 & e & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & 0 & d & 0 \\
0 & 0 & q & 0 & g & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{1.3}$$

where

$$a=\frac{q(x-1)}{q^2x-1},\qquad b=\frac{q^2(qx+1)(x-1)}{(q^2x-1)(q^3x+1)},\qquad c=\frac{q(x-1)}{q^2x-1}+\frac{(q^2-1)(q^3+1)x}{(q^2x-1)(q^3x+1)},$$
$$d=\frac{q^2-1}{(q^2x-1)},\qquad e=\frac{(q^2-1)(x-1)q^{1/2}}{(q^2x-1)(q^3x+1)},\qquad p=\frac{(q^2-1)(q^3x+qx-q+1)}{(q^2x-1)(q^3x+1)},$$
$$f=\frac{(q^2-1)x}{(q^2x-1)},\qquad g=\frac{(1-q^2)x(x-1)q^{5/2}}{(q^2x-1)(q^3x+1)},\qquad q=\frac{(q^2-1)x(q^3x-q^2x+q^2+1)}{(q^2x-1)(q^3x+1)}.$$

We consider the generating functions

$$l_{ij}^{\pm}(z)=\sum_{n=0}^{\infty}l_{ij}^{\pm}[\pm n]z^{\mp n},\qquad 1\leqslant i,j\leqslant 3$$
$$l_{ij}^{+}[0]=l_{ji}^{-}[0]=0,\qquad 1\leqslant i<j\leqslant 3$$

and the shifted generating functions

$$\tilde{l}_{ij}^{\pm}(z)=q^{\frac{1}{2}(i-j)}l_{4-j,4-i}^{\pm}(-q^{-3}z).$$

We define the *L-operators* $L^{\pm}(z)$ and their shifts $\tilde{L}^{\pm}(z)$ as

$$L^{\pm}(z)=\left(l_{ij}^{\pm}(z)\right)_{i,j=1}^{3},\qquad \tilde{L}^{\pm}(z)=\left(\tilde{l}_{ij}^{\pm}(z)\right)_{i,j=1}^{3}.$$

Finally, we recall the notion of q-determinant:

$$\det_q(L^{\pm}(z))=\sum_{\tau\in\mathfrak{S}_3}(-q)^{\mathrm{sgn}(\tau)}l_{1\tau(1)}^{\pm}(z)l_{2\tau(2)}^{\pm}(q^2z)l_{3\tau(3)}^{\pm}(q^4z),$$

where $\mathfrak{S}_n$ is the permutation group on n elements.

Define $\mathcal{A}_{\mathcal{R}}$ as the associative algebra with the generators $l_{ij}^{\pm}[\pm n]$, $n\in\mathbb{N}$, $1\leqslant i,j\leqslant 3$ and $l_{ij}^{+}[0]$, $l_{ji}^{-}[0]$ for $1\leqslant j\leqslant i\leqslant 3$, subject to the following relations:

$$\det_q(L^{\pm}(z))=1, \tag{1.4}$$

$$L^{\pm}(z)\tilde{L}^{\pm}(z)=I_3, \tag{1.5}$$

$$l_{ii}^{+}[0]l_{ii}^{-}[0]=l_{ii}^{-}[0]l_{ii}^{+}[0]=1, \tag{1.6}$$

$$\begin{aligned} R\left(\frac{z}{w}\right)L_1^{\pm}(z)L_2^{\pm}(w)&=L_2^{\pm}(w)L_1^{\pm}(z)R\left(\frac{z}{w}\right),\\ R\left(\frac{z}{w}\right)L_1^{+}(z)L_2^{-}(w)&=L_2^{-}(w)L_1^{+}(z)R\left(\frac{z}{w}\right). \end{aligned} \tag{1.7}$$

where $L_1(z)=L(z)\otimes I_3$, $L_2(z)=I_3\otimes L(z)$, and I_n denotes the $n\times n$ identity matrix.

The Hopf algebra structure on $\mathcal{A}_{\mathcal{R}}$ is given by

$$\Delta_R(l_{ij}^{\pm})=\sum_{k=1}^{3}l_{ik}^{\pm}\otimes l_{kj}^{\pm},\qquad S_R(L^{\pm})=(L^{\pm})^{-1},\qquad \varepsilon_R(L^{\pm})=I_3,$$

where Δ_R, S_R and ε_R are the comultiplication, the antipode and the counit maps, respectively.

The L-operators admit the Gaussian decomposition

$$L^{\pm}(z) = \begin{pmatrix} 1 & f_1^{\pm}(z) & f_{13}^{\pm}(z) \\ 0 & 1 & f_2^{\pm}(z) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_1^{\pm}(z) & 0 & 0 \\ 0 & k_2^{\pm}(z) & 0 \\ 0 & 0 & k_3^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ e_1^{\pm}(z) & 1 & 0 \\ e_{31}^{\pm}(z) & e_2^{\pm}(z) & 1 \end{pmatrix}, \tag{1.8}$$

where

$$e_i^{\pm}(z) = \sum_{n\geqslant 0} e_i^{\pm}[n] z^{\mp n}, \quad i = 1, 2, \qquad e_{31}^{\pm}(z) = \sum_{n\geqslant 0} e_{31}^{\pm}[n] z^{\mp n},$$

$$f_i^{\pm}(z) = \sum_{n\geqslant 0} f_i^{\pm}[n] z^{\mp n}, \quad i = 1, 2, \qquad f_{13}^{\pm}(z) = \sum_{n\geqslant 0} f_{13}^{\pm}[n] z^{\mp n},$$

$$k_i^{\pm}(z) = \sum_{n\geqslant 0} k_i^{\pm}[n] z^{\mp n}, \quad i = 1, 2, 3.$$

and

$$e_1^{-}[0] = e_2^{-}[0] = e_{31}^{-}[0] = 0, \qquad f_1^{+}[0] = f_2^{+}[0] = f_{13}^{+}[0] = 0.$$

We have thus defined the Gaussian generators

$$f_1^{\pm}[n],\ f_2^{\pm}[n],\ f_{13}^{\pm}[n], \qquad e_1^{\pm}[n],\ e_2^{\pm}[n],\ e_{31}^{\pm}[n], \qquad k_1^{\pm}[n],\ k_2^{\pm}[n],\ k_3^{\pm}[n], \quad n \geqslant 0,$$

subject to relations (1.4) – (1.7). Here $f_1^{\pm}[n]$, $f_2^{\pm}[n]$, $e_1^{\pm}[n]$, $e_2^{\pm}[n]$ form the "minimal" set of generators, while $f_{13}^{\pm}[n]$ and $e_{31}^{\pm}[n]$ form the extended one.

2 Links between realizations

2.1 Universal $\mathcal{R}$-matrix

Recall that the universal $\mathcal{R}$-matrix of a quasitriangular Hopf algebra $\mathcal{A}$ is an element $\mathcal{R} \in \mathcal{A}\otimes\mathcal{A}$, subject to the following relations:

$$\Delta^{op}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \tag{2.1}$$

for any $x \in \mathcal{A}$, and

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{2.2}$$

where

$$\mathcal{R}_{12} = \mathcal{R} \otimes 1, \qquad \mathcal{R}_{23} = 1 \otimes \mathcal{R}, \qquad \mathcal{R}_{13} = (\sigma \otimes \mathrm{id})(\mathcal{R}_{23})$$

are elements of $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

Now we fix the representation π_z defined on Chevalley generators as

$$\pi_z(e_\alpha) = \begin{pmatrix} 0 & q^{\frac{1}{4}} & 0 \\ 0 & 0 & -q^{-\frac{1}{4}} \\ 0 & 0 & 0 \end{pmatrix}, \qquad \pi_z(e_{\delta-2\alpha}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{q+q^{-1}} & 0 & 0 \end{pmatrix} z,$$

$$\pi_z(e_{-\alpha}) = \begin{pmatrix} 0 & 0 & 0 \\ q^{-\frac{1}{4}} & 0 & 0 \\ 0 & -q^{\frac{1}{4}} & 0 \end{pmatrix}, \qquad \pi_z(e_{-\delta+2\alpha}) = \begin{pmatrix} 0 & 0 & \sqrt{q+q^{-1}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} z^{-1},$$

$$\pi_z(k_\alpha) = \begin{pmatrix} q & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-1} \end{pmatrix}, \qquad \pi_z(k_{\delta-2\alpha}) = \begin{pmatrix} q^{-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{2} \end{pmatrix}.$$

After the representation is chosen, we obtain the R-matrix [3]

$$R\left(\frac{z}{w}\right) = (\pi_z \otimes \pi_w)\,\mathcal{R}$$

directly from the definition of $\mathcal{R}$ by applying $\pi_z \otimes \pi_w$ to relation (2.1) and substituting the Chevalley generators for x. Condition (2.1) defines $R(x)$ up to a scalar.

2.2 Chevalley and Drinfeld realizations

Following [KT], we represent the universal $\mathcal{R}$-matrix of $\mathcal{A}_{\mathcal{C}h}$ as [4]:

$$\begin{aligned} \mathcal{R} &= \mathcal{R}_- \mathcal{K} q^{-h\otimes h} \mathcal{R}_+, & \mathcal{R}_- &= \left(\prod_{\gamma<\delta}^{\rightarrow} \exp_{q_\gamma}\left((q^{-1}-q)e_\gamma \otimes e_{-\gamma}\right)\right), \\ \mathcal{K} &= \exp\left(-\sum_{n>0} S_n\right), & \mathcal{R}_+ &= \left(\prod_{\gamma>\delta}^{\leftarrow} \exp_{q_\gamma}\left((q^{-1}-q)e_\gamma k_\alpha^{-(\alpha,\gamma)} \otimes k_\alpha^{(\alpha,\gamma)} e_{-\gamma}\right)\right), \end{aligned} \tag{2.3}$$

where γ ranges over real positive roots, h is defined by

$$q^{\pm h} = k^{\pm 1},$$

and S_n is given by the formula

$$S_n = \frac{n\left(q-q^{-1}\right)^2\left(q+1+q^{-1}\right)\left(e_{n\delta}\otimes e_{-n\delta}\right)}{\left(q^n-q^{-n}\right)\left(q^n+(-1)^{n+1}+q^{-n}\right)}.$$

Now, we can find the scalar prefactor for $R(x)$. Applying $\pi_z \otimes \pi_w$ to $\mathcal{K}\cdot q^{-h\otimes h}$, we obtain the scalar

$$q^{-1}\exp\left(-\sum_{n>0}\frac{\left(q^n-q^{-n}\right)x^n}{\left(q^n+(-1)^{n+1}+q^{-n}\right)n}\right). \tag{2.4}$$

Therefore, $R(x)$ is equal to matrix (1.3) times scalar prefactor (2.4).

Remark 2.1 *Scalar (2.4) expands into the product*

$$q^{-1}\prod_{k=0}^{\infty}\left(\frac{(1+q^{3k+2}x)(1-q^{3k+3}x)}{(1+q^{3k}x)(1-q^{3k+1}x)}\right)^{(-1)^k},$$

in the domain $|q|<1$ and into the product

$$q^{-1}\prod_{k=0}^{\infty}\left(\frac{(1+q^{-3k}x)(1-q^{-(3k+1)}x)}{(1+q^{-(3k+2)}x)(1-q^{-(3k+3)}x)}\right)^{(-1)^k},$$

in the domain $|q|>1$.

[3] The scaling of the basis of the $U_q(A_2^{(2)})$-module is chosen in such a way that R-matrix $R(x)$ becomes symmetric with respect to the secondary diagonal. In that basis, the R-matrix coincides with the one obtained in [J].

[4] The formula for the $\mathcal{R}$-matrix differs from the formula in [KT] because q-commutators are replaced with q^{-1}-commutators. Moreover, our formula differs from the one in [KS]; the $\mathcal{R}$-matrix in [KS] is inverse to the $\mathcal{R}$-matrix here.

Theorem 1 *The isomorphism between associative algebras $\mathcal{A}_{\mathcal{C}h}$ and $\mathcal{A}_{\mathcal{D}}$ is established by the following map:*

$$\begin{array}{rclcrcl} k_{\delta-2\alpha} & \mapsto & k^{-2}, & \qquad & k_\alpha & \mapsto & k, \\ e_{\delta-2\alpha} & \mapsto & a(qf_1f_0 - f_0f_1)k^{-2}, & & e_\alpha & \mapsto & e_0, \\ e_{-\delta+2\alpha} & \mapsto & ak^2(q^{-1}e_0e_{-1} - e_{-1}e_0), & & e_{-\alpha} & \mapsto & f_0, \end{array} \tag{2.5}$$

where a is defined in (1.2).

Proof: A straightforward verification and the inductive construction of the Cartan-Weyl basis show that the map (2.5) above is a surjective homomorphism. More precisely, every Drinfeld generator can be obtained as

$$\begin{array}{rclll rclll} e_{\alpha+n\delta} & \mapsto & e_n, & n \geqslant 0, & \qquad & -k_\alpha^{-1}e_{\alpha-n\delta} & \mapsto & e_{-n}, & n > 0, \\ e_{-\alpha-n\delta} & \mapsto & f_{-n}, & n \geqslant 0, & & -e_{-\alpha+n\delta}k_\alpha & \mapsto & f_n, & n > 0, \\ k_\alpha & \mapsto & k, & & & -b^{-1}e_{n\delta} & \mapsto & a_n, & n \in \mathbb{Z} \setminus \{0\}. \end{array} \tag{2.6}$$

Reversing all the arrows in map (2.6), we obtain a map inverse to (2.5). Finally, map (2.6) with reversed arrows is also an epimorphism, and the statement of the theorem follows. □

Remark 2.2 *The above map is an isomorphism of associative algebras. It preserves the counit but does not respect comultiplication maps Δ and $\Delta_{\mathcal{D}}$.*

As for the link between the Drinfeld and the standard comultiplications, we have the following

Proposition 2.1 *The tensor $\mathcal{R}_-$ is a cocycle for $\Delta_{\mathcal{D}}$ so that for any $x \in \mathcal{A}_{\mathcal{C}h}$*

$$\Delta(x) = (\mathcal{R}_-)^{-1}\, \Delta_{\mathcal{D}}(x)\mathcal{R}_-.$$

The proof is given in [KS]. Another expression for $\mathcal{R}_-$ in terms of the Drinfeld generators can also be found there.

2.3 From Chevalley to RLL realization

According to [D], the universal R-matrix of any quasitriangular Hopf algebra satisfies the quantum Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{2.7}$$

Theorem 2 *An isomorphism between the associative algebras $\mathcal{A}_{\mathcal{R}}$ and $\mathcal{A}_{\mathcal{C}h}$ can is given by the maps*

$$\begin{aligned} L^-(z) &\mapsto (\pi_z \otimes \mathrm{id})\mathcal{R}, \\ L^+(z) &\mapsto (\pi_z \otimes \mathrm{id})\mathcal{R}_{21}^{-1}. \end{aligned} \tag{2.8}$$

Proof: Applying $\pi_z \otimes \pi_w \otimes \mathrm{id}$, $\pi_z \otimes \mathrm{id} \otimes \pi_w$ and $\mathrm{id} \otimes \pi_z \otimes \pi_w$ to relation (2.7), we derive commutation relations (1.6) and (1.7). Using formulas (2.3) and (2.8), we obtain the Gaussian decomposition (1.8).

The matrix $R(-q^3)$ has a one-dimensional kernel, which is spanned by vector

$$v = q^{-\frac{1}{2}}v_1 \otimes v_3 + v_2 \otimes v_2 + q^{\frac{1}{2}}v_3 \otimes v_1,$$

where $\{v_1, v_2, v_3\}$ is the basis of the representation π_z. It follows that v is an eigenvector of operator $L_1(z)L_2(-q^{-3}z)$. Let $\lambda(z)$ be an eigenvalue of v. One can check that $\left(\pi_z \otimes \pi_{-q^{-3}z}\right)\Delta(x)$ vanishes on v for every $x \in \mathcal{A}_{\mathcal{C}h}$ (it is sufficient to varify the vanishing condition for the Chevalley generators only). Applying $\left(\pi_z \otimes \pi_{-q^{-3}z} \otimes \mathrm{id}\right)$ to the first equation in (2.2), we obtain $\lambda(z) = 1$. Hence, v is stable under $L_1(z)L_2(-q^{-3}z)$, which turns out to be equivalent to the relation (1.5):

$$\begin{pmatrix} l_{11}(z) & l_{12}(z) & l_{13}(z) \\ l_{21}(z) & l_{22}(z) & l_{23}(z) \\ l_{31}(z) & l_{32}(z) & l_{33}(z) \end{pmatrix} \begin{pmatrix} l_{33}(-q^{-3}z) & q^{-\frac{1}{2}}l_{23}(-q^{-3}z) & q^{-1}l_{13}(-q^{-3}z) \\ q^{\frac{1}{2}}l_{32}(-q^{-3}z) & l_{22}(-q^{-3}z) & q^{-\frac{1}{2}}l_{12}(-q^{-3}z) \\ ql_{31}(-q^{-3}z) & q^{\frac{1}{2}}l_{21}(-q^{-3}z) & l_{11}(-q^{-3}z) \end{pmatrix} = I_3.$$

Next, it is possible to check that

$$\det_q(L^{\pm}(z)) = k_1^{\pm}(z)k_2^{\pm}(q^{-2}z)k_3^{\pm}(q^{-4}z).$$

Following the procedure of the construction of the Cartan-Weyl basis we get

$$\pi_z\left(e_{\pm n\delta}\right) = b\frac{q^n - q^{-n}}{n(q-q^{-1})}\begin{pmatrix} -z^{\pm n} & 0 & 0 \\ 0 & (q^2z)^{\pm n} - (-qz)^{\pm n} & 0 \\ 0 & 0 & (-q^3z)^{\pm n} \end{pmatrix}, \quad n > 0.$$

Applying π_z to either first or second tensor component of $\mathcal{K}$, we arrive at

$$\begin{aligned} k_1^{\pm}(z) &= k_3^{\pm}(-q^{-3}z)^{-1}, \\ k_2^{\pm}(z) &= k_3^{\pm}(-q^{-1}z)k_3^{\pm}(q^{-2}z)^{-1}, \end{aligned} \tag{2.9}$$

which implies the desired relation (1.4).

Since all the Chevalley generators are in the image of map (2.8), we obtain an epimorphism from $\mathcal{A}_{\mathcal{R}}$ to $\mathcal{A}_{\mathcal{C}h}$ (and, therefore, to $\mathcal{A}_{\mathcal{D}}$). In Theorem 3 below, we establish an epimorphism from $\mathcal{A}_{\mathcal{D}}$ to $\mathcal{A}_{\mathcal{R}}$, which is inverse to (2.8). Hence, the theorem is proved modulo the result of Theorem 3. □

Proposition 2.2 *Let $\varphi(x) \in \mathcal{A}_{\mathcal{C}h}$ be the image of $x \in \mathcal{A}_{\mathcal{R}}$ under isomorphism (2.8). Then*

$$\varphi\left(\Delta_R(x)\right) = \sigma \circ \Delta(\varphi(x)).$$

Proof: The proof can be obtained by applying $\pi_z \otimes \mathrm{id} \otimes \mathrm{id}$ to the relation

$$(\mathrm{id} \otimes \Delta^{op})\mathcal{R} = \mathcal{R}_{12}\mathcal{R}_{13},$$

which is equivalent to (2.2). □

2.4 From RLL to Drinfeld realization

First of all, we express all generators of the RLL realization in terms of $e_1^{\pm}(z)$, $f_1^{\pm}(z)$, $k_1^{\pm}(z)$. The expressions

$$\begin{aligned} k_2^{\pm}(z) &= k_1^{\pm}(-qz)k_1^{\pm}(q^2z)^{-1}, \\ k_3^{\pm}(z) &= k_1^{\pm}(-q^3z)^{-1} \end{aligned} \tag{2.10}$$

follow directly from (2.9). Relations (1.7) imply

$$(q^2-1)z(q^3z+w)L^{\pm}_{23}(z)L^{\pm}_{33}(w)+q(z-w)(q^3z+w)L^{\pm}_{33}(z)L^{\pm}_{23}(w)=\\ =(q^2z-w)(q^3z+w)L^{\pm}_{23}(w)L^{\pm}_{33}(z),$$

which results in

$$q(z-w)k^{\pm}_3(z)f^{\pm}_2(w)=(q^2z-w)f^{\pm}_2(w)k^{\pm}_3(z)-(q^2-1)zf^{\pm}_2(z)k^{\pm}_3(z).$$

Setting $w=q^2z$, we get

$$f^{\pm}_2(q^2z)=q^{-1}k^{\pm}_3(z)^{-1}f^{\pm}_2(z)k^{\pm}_3(z).$$

On the other hand, relation (1.5) implies

$$f^{\pm}_1(z)=-q^{-\frac{1}{2}}k^{\pm}_3(-q^{-3}z)^{-1}f^{\pm}_2(-q^{-3}z)k^{\pm}_3(-q^{-3}z).$$

We hence obtain

$$\begin{aligned} f^{\pm}_2(z)&=-q^{-\frac{1}{2}}f^{\pm}_1(-qz),\\ e^{\pm}_2(z)&=-q^{\frac{1}{2}}e^{\pm}_1(-qz), \end{aligned} \tag{2.11}$$

where the expression for $e^{\pm}_2(z)$ is derived in the similar way. The expressions for $e^{\pm}_{31}(z)$ and $f^{\pm}_{13}(z)$ are considered in Section 3.

Remark 2.3 *It is also possible to use the explicit formula for $L^{\pm}(z)$ to derive relations (2.11), as we did for relations (2.9).*

Now let

$$\begin{aligned} E_i(z)&=e^{+}_i(z)-e^{-}_i(z),\\ F_i(z)&=f^{+}_i(z)-f^{-}_i(z). \end{aligned}$$

Then the equalities

$$\begin{aligned} E_2(z)&=-q^{\frac{1}{2}}E_1(-qz),\\ F_2(z)&=-q^{-\frac{1}{2}}F_1(-qz). \end{aligned} \tag{2.12}$$

hold. Moreover, relations (1.4)–(1.7) imply

$$\begin{aligned} k^{\pm}_1(z)k^{\pm}_1(w)&=k^{\pm}_1(w)k^{\pm}_1(z),\\ k^{-}_1(z)k^{+}_1(w)&=k^{+}_1(w)k^{-}_1(z), \end{aligned} \tag{2.13}$$

$$\begin{aligned} k^{\pm}_1(z)E_1(w)k^{\pm}_1(z)^{-1}&=\frac{q(z-w)}{q^2z-w}E_1(w),\\ k^{\pm}_1(z)^{-1}F_1(w)k^{\pm}_1(z)&=\frac{z-q^2w}{q(z-w)}F_1(w), \end{aligned} \tag{2.14}$$

$$\begin{aligned} (qz+w)(z-q^2w)E_1(z)E_1(w)&=(z+qw)(q^2z-w)E_1(w)E_1(z),\\ (z+qw)(q^2z-w)F_1(z)F_1(w)&=(qz+w)(z-q^2w)F_1(w)F_1(z), \end{aligned} \tag{2.15}$$

$$[E_1(z),F_1(w)]=(q-q^{-1})\delta\left(\frac{z}{w}\right)\left(k^{+}_1(w)k^{+}_2(w)^{-1}-k^{-}_1(z)k^{-}_2(z)^{-1}\right). \tag{2.16}$$

Theorem 3 *An isomorphism*[5] *between the algebras* $\mathcal{A}_{\mathcal{D}}$ *and* $\mathcal{A}_{\mathcal{R}}$ *is given by the following maps:*

$$\begin{array}{lcl} q^{-\frac{1}{4}}\left(q-q^{-1}\right)e(qz) & \mapsto & E_1(z), \\ q^{\frac{1}{4}}\left(q-q^{-1}\right)f(qz) & \mapsto & F_1(z), \\ K^{\pm}(qz) & \mapsto & k_1^{\pm}(z)k_2^{\pm}(z)^{-1}. \end{array} \tag{2.17}$$

Proof: Formulas (2.13)–(2.16) imply that the map (2.17) is a homomorphism. For all $n \geqslant 0$ Gaussian coordinates $k_1^{\pm}[n]$ can be obtained step by step from the product $k_1^{\pm}(z)k_2^{\pm}(z)^{-1}$, which together with relations (2.10) and (2.12) imply that map (2.17) is an epimorphism. Finally, one can check that the composition of maps (2.8), (2.5) and (2.17) is the identity map. $\square$

3 Link with composite Drinfeld currents

Here we revise some definitions, given in [KS] and then express the generating functions $f_{13}^{\pm}(z)$ and $e_{31}^{\pm}(z)$ in terms of the Drinfeld currents.

Recall that in any quantum affine algebra there exist two types of Borel subalgebras. Borel subalgebras of the first type come from the Drinfeld realization. Let U_F denote the subalgebra of $U_q(A_2^{(2)})$ generated by $k^{\pm 1}, f_n$, $n \in \mathbb{Z}$; a_n, $n > 0$, and let U_E denote the subalgebra of $U_q(A_2^{(2)})$ generated by $k^{\pm 1}, e_n$, $n \in \mathbb{Z}$; a_n, $n < 0$. The "current" Borel subalgebra U_F contains subalgebra U_f generated by f_n, $n \in \mathbb{Z}$. The "current" Borel subalgebra U_E contains subalgebra U_e generated by e_n, $n \in \mathbb{Z}$.

Borel subalgebras of the second type are obtained via the Chevalley realization. Let $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$ denote a pair of subalgebras of $U_q(A_2^{(2)})$ generated by

$$e_\alpha,\ e_{\delta-2\alpha},\ k_\alpha^{\pm 1} \qquad \text{and} \qquad e_{-\alpha},\ e_{-(\delta-2\alpha)},\ k_\alpha^{\pm 1}$$

respectively. In terms of the Drinfeld realization, these subalgebras are generated by

$$k^{\pm 1},\ e_0,\ qf_1f_0 - f_0f_1 \qquad \text{and} \qquad k^{\pm 1},\ f_0,\ q^{-1}e_0e_{-1} - e_{-1}e_0.$$

respectively.

Let U_F^+, U_f^-, U_e^+ and U_E^- denote the following intersections of Borel subalgebras:

$$\begin{array}{ll} U_f^- = U_F \cap U_q(\mathfrak{b}_-), & U_F^+ = U_F \cap U_q(\mathfrak{b}_+), \\ U_e^+ = U_E \cap U_q(\mathfrak{b}_+), & U_E^- = U_E \cap U_q(\mathfrak{b}_-). \end{array}$$

The superscript sign indicates the Borel subalgebra $U_q(\mathfrak{b}_\pm)$ containing the given algebra, and the subscript letter indicates "current" Borel subalgebra U_F or U_E that it is intersected with. These letters are capitals if the subalgebra contains imaginary root generators a_n and are lower case otherwise.

Let P^+ and P^- denote projection operators such that for any $f_+ \in U_F^+$ and for any $f_- \in U_f^-$

$$P^+(f_-f_+) = \varepsilon(f_-)f_+, \qquad P^-(f_-f_+) = f_-\varepsilon(f_+). \tag{3.1}$$

[5] As in remark (2.2), the mapping above is an isomorphism of associative algebras. It preserves the counit but does not respect comultiplication maps Δ and $\Delta_{\mathcal{D}}$.

Also let P^{*+} and P^{*-} denote projecton operators such that for any $e_+ \in U_e^+$ and for any $e_- \in U_E^-$

$$P^{*+}(e_+e_-) = e_+\varepsilon(e_-), \qquad P^{*-}(e_+e_-) = \varepsilon(e_+)e_-. \tag{3.2}$$

Finally, let us introduce composite currents $s(z)$ and $r(z)$ as follows:

$$\begin{aligned} s(z) &= \operatorname*{res}_{w=-q^{-1}z} f(z)f(w)\frac{dw}{w}, \\ r(z) &= \operatorname*{res}_{w=-qz} e(w)e(z)\frac{dw}{w}. \end{aligned}$$

Remark 3.1 *The currents $s(z)$ and $\tilde{s}(z)$ were defined in [KS]. The definition of $s(z)$ coincides with the one given above, while $\tilde{s}(z) = -s(-qz)$ or equivalently*

$$\tilde{s}(z) = \operatorname*{res}_{w=-qz} f(w)f(z)\frac{dw}{w}.$$

An involution ι of the algebra $U_q(A_2^{(2)})$ was also defined in [KS]:

$$\iota(e_n) = f_{-n}, \qquad \iota(f_n) = e_{-n}, \qquad \iota(a_n) = a_{-n}, \qquad \iota(K_0) = K_0^{-1}.$$

Therefore, the current $r(z)$ satisfies the equality

$$r(z) = \iota\left(-\tilde{s}\left(z^{-1}\right)\right).$$

Proposition 3.1 *Isomorphism (2.17) maps the Gaussian coordinates $e_{31}^{\pm}(z)$ and $f_{13}^{\pm}(z)$ to the expressions*

$$\begin{aligned} e_{31}^{\pm}(z) &\mapsto \mp(1-q)\left(q^{-1}\left[P^{*\pm}(e(-q^2z)), e_0\right]_{q^{-1}} + \left[-q^2ze_{-1}, P^{*\pm}(e(-q^2z)) \mp e_0\right]_{q^{-1}}\right), \\ f_{13}^{\pm}(z) &\mapsto \pm(1-q)\left(q\left[P^{\pm}(f(-q^2z)), f_0\right]_{q^{-1}} + \left[(-q^2z)^{-1}f_1, P^{\pm}(f(-q^2z)) \pm f_0\right]_{q^{-1}}\right). \end{aligned} \tag{3.3}$$

Proof: We consider only the case of $f_{13}^+(z)$ because the other cases are similar. Relations (1.7) imply

$$\begin{aligned} &(q^2-1)z(q^3z+w)l_{13}^+(z)l_{22}^-(w) + q(z-w)(q^3z+w)l_{23}^+(z)l_{12}^-(w) = \\ &\qquad = (q^2-1)w(q^3z+w)l_{13}^-(w)l_{22}^+(z) + q(z-w)(q^3z+w)l_{12}^-(w)l_{23}^+(z). \end{aligned}$$

Assuming $w=0$, we get

$$(q-q^{-1})l_{13}^+(z) = \left[l_{12}^-(0), l_{23}^+(z)\right],$$

or, equivalently,

$$(q-q^{-1})f_{13}^+(z)k_3^+(z) = f_1^-[0]f_2^+(z)k_3^+(z) - f_2^+(z)k_3^+(z)f_1^-[0].$$

Using (1.7) once again, we obtain

$$k_3^+(z)f_1^-[0] = qf_1^-[0]k_3^+(z) + (q-q^{-1})q^{\frac{1}{2}}f_2^+(z)k_3^+(z),$$

and therefore derive

$$f_{13}^+(z) = \frac{1}{q-q^{-1}}\left(q^{\frac{1}{2}}f_1^+(-qz)f_1^-[0] - q^{-\frac{1}{2}}f_1^-[0]f_1^+(-qz)\right) - q^{-\frac{1}{2}}f_1^+(-qz)^2.$$

Now, let us recall a result from [KS]: on one hand, we have

$$P(f(z_1)f(z_2)) = f^+(z_1)\left(f^+(z_2) - \frac{q^2-1}{q^2-z_2/z_1}f^+(z_1)\right) - \\ - \frac{(1+q^3)(1-z_2/z_1)}{(1+q)(q^2-z_2/z_1)(1+qz_2/z_1)}P(s(z_1)),$$

and, on the other hand,

$$P(f(z_1)f(z_2)) = \frac{(q^2-z_1/z_2)(q^{-1}+z_1/z_2)}{(1-q^2z_1/z_2)(1+q^{-1}z_1/z_2)}P(f(z_2)f(z_1)).$$

Equating the right-hand sides of these expressions, multiplying them by z_2, and letting z_2 tend to infinity, we obtain

$$(q-q^{-1)}P(f(z))^2 = \left(f_1P(f(z)) - q^{-1}P(f(z))f_1\right)z^{-1} + \frac{1+q^3}{q(1+q)}P(s(z)) + \left(f_1f_0 - q^{-1}f_0f_1\right)z^{-1}.$$

Finally, since isomorphism (2.17) maps $f_1^{\pm}(-qz)$ to $\pm q^{\frac{1}{4}}(q-q^{-1})P^{\pm}(f(-q^2z))$, we derive the desired formula (3.3). □

Theorem 4 *Isomorphism (2.17) links the Gaussian coordinates $f_{13}^+(z)$, $e_{31}^+(z)$ and the composite Drinfeld currents $s(z)$, $r(z)$ by the following expressions:*

$$\begin{aligned}
P^+(s(-q^2z)) &= \frac{1}{(q-q^{-1})(q-1+q^{-1})}f_{13}^+(z),\\
P^-(s(-q^2z)) &= \frac{1}{(q-q^{-1})(q-1+q^{-1})}\left(f_1^-(z)f_2^-(z) - f_{13}^-(z)\right),\\
P^{*+}(r(-q^2z)) &= \frac{1}{(q-q^{-1})(q-1+q^{-1})}e_{31}^+(z),\\
P^{*-}(r(-q^2z)) &= \frac{1}{(q-q^{-1})(q-1+q^{-1})}\left(e_2^-(z)e_1^-(z) - e_{31}^-(z)\right).
\end{aligned} \tag{3.4}$$

Proof: The statement for $s(z)$ follows directly from Proposition 3.1 and Theorem 1 in [KS]. For $r(z)$ the involution ι should also be used. □

Remark 3.2 *Since $f_1^-(z)f_2^-(z) - f_{13}^-(z)$ and $e_2^-(z)e_1^-(z) - e_{31}^-(z)$ are the corner elements in the Gaussian decomposition of the inverse L-operator $\left(L^-(z)\right)^{-1}$, it might make sense to use the elements of the decomposition of the inverse L-operator $\left(L^-(z)\right)^{-1}$ as the Gaussian coordinates instead of the elements of the decomposition of usual L-operator $L^-(z)$.*

Acknowledgements

The author thanks Sergey Khoroshkin for the invaluable help and his constant interest to the work. The author is also grateful to Stanislav Pakuliak for many fruitful discussions. The author was supported by RFBR grant 08-01-00667, joint CNRS-RFBR grant 09-01-93106-NCNIL, and by Federal Agency for Science and Innovations of Russian Federation under contract 14.740.11.0081.

References

[CP] V. Chari, A. Pressley. Twisted quantum affine algebras. *Comm. Math. Phys.* 198 (1998), No.2, 461–476.

[D] V. Drinfeld. New realization of Yangians and quantum affine algebras. *Sov. Math. Dokl.* 36 (1988), 212–216.

[DF] J. Ding, I. Frenkel. Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}(n)})$. *Comm. Math. Phys.* 156 (1993), No.2, 212–216.

[DK] J. Ding, S. Khoroshkin. Universal R-matrix for quantum affine algebras $U_q(A_2^{(2)})$ and $U_q(\widehat{osp}(1|2))$ with Drinfeld comultiplication. *Adv. in Math.* 189 (2004), 413–438.

[DK2] J. Ding, S. Khoroshkin. On the FRTS approach to quantized current algebras. *Lett. Math. Phys.* 45 (1998), No.4, 331–352.

[DK3] J. Ding, S. Khoroshkin. Weyl group extension of quantized current algebras. *Transform. Groups* 5 (2000), pp.35-59.

[EKP] B. Enriquez, S. Khoroshkin, S. Pakuliak. Weight functions and Drinfeld currents. *Comm. Math. Phys.* 276 (2007), No.3, 691–725.

[FRS] D. Fioravanti, M. Stanishkov, F. Ravanini. Generalized KdV and Quantum Inverse Scattering description of Conformal Minimal Models. *Phys. Lett. B.* 367 (1996), 113–120.

[FRT] L. Faddeev, N. Reshetikhin, L. Takhtajan. Quantization of Lie groups and Lie algebras. *Leningrad Math. J.* 1 (1990), 193–225.

[FR] I. Frenkel, N. Reshetikhin. Quantum affine algebras and holonomic difference equations. . *Comm. Math. Phys.* 146 (1992), 1–60.

[H] D. Hernandez. Kirillov-Reshetikhin conjecture: the general case. *Int. Math. Res. Not.* 2010, No.1, 149–193.

[IK] A. Izergin, V. Korepin. The inverse scattering method approach to the quantum Shabat-Mikhailov model. *Comm. Math. Phys.* 79 (1981), No.3, 303–316.

[J] M. Jimbo. Quantum R-matrix for the generalized Toda system. *Comm. Math. Phys.* 102 (1986), No.4, 537–547.

[KP] S. Khoroshkin, S. Pakuliak. A computation of an universal weight function for the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ *J. Math. Kyoto Univ.* 48 (2008), 277-322.

[KS] S. Khoroshkin, A. Shapiro. Weight function for the quantum affine algebra $U_q(A_2^{(2)})$. *Geom. and Phys.* 60 (2010), 1833-1851.

[KT] S. Khoroshkin, V. Tolstoy. The uniqueness theorem for the universal R-matrix. *Let. Math. Phys.* 24 (1992), No.3, 231–244.

[RS] N. Reshetikhin, M. Semenov-Tian-Shansky. Central extensions of quantum current groups. *Lett. Math. Phys.* 19 (1990), 133–142.

[T] V. Tarasov. Algebraic Bethe ansatz for the Izergin-Korepin R-matrix. *Theor. and Math. Phys.* 76 (1988), No.2, 793-803.

[YZ] W. Yang, Y. Zhang. Drinfeld basis of the twisted quantum affine algebra $U_q(A_2^{(2)})$ from the Gauss decomposition of an L-operator. *J. Phys. A: Math. Gen.* 34 (2001), L205.